PAUTA CONTROL 2 - MA11A-ALGEBRA

(1998)

Pregunta 1.

(i)

$$\sum_{i=0}^{n-1} \frac{(p+i)!}{i!} = \frac{1}{(p+1)} \frac{(p+n)!}{(n-1)!} \quad (*)$$

$\underline{n=1}$: $\sum_{i=0}^{1-1} \frac{(p+i)!}{i!} = \frac{p!}{0!} = p!$

y

$$\frac{1}{(p+1)} \cdot \frac{(p+1)!}{(1-1)!} = p! \ . \ \text{Luego (*) es cierta para n=1.}$$

H.I.: asumimos (*) para n $\geq$ 1.

Probamos el caso $\underline{n+1}$ asumiendo (*):

$$\sum_{i=0}^{n} \frac{(p+i)!}{i!} = \sum_{i=0}^{n-1} \frac{(p+i)!}{i!} + \frac{(p+n)!}{n!}$$

$$\begin{aligned}
&\text{(H.I.)} \\
&= \frac{1}{(p+1)} \cdot \frac{(p+n)!}{(n-1)!} + \frac{(p+n)!}{n!} \\
&= \frac{(p+n)!}{(n-1)!} \left[\tfrac{1}{p+1} + \tfrac{1}{n} \right] \\
&= \frac{(p+n)!}{(n-1)!} \frac{(p+n+1)}{n(p+1)} \\
&= \frac{(p+n+1)!}{n!} \cdot \frac{1}{(p+1)} \ \text{que prueba (*) para n+1}
\end{aligned}$$

Para calcular $\sum_{k=1}^{n} k$ hay que considerar p = 1. En efecto,

$$\sum_{i=0}^{n-1} \frac{(1+i)!}{i!} = \sum_{i=0}^{n-1} (i+1) = \sum_{i=1}^{n} i = \frac{1}{2} \cdot \frac{(n+1)!}{(n-1)!} = \frac{1}{2} n(n+1)$$

Para calcular $\sum\limits_{k-1}^{n} k^2$ evaluemos en p = 2:

$$\sum_{i=0}^{n-1} \frac{(i+2)!}{i!} = \sum_{i=0}^{n-1} (i+1)\ (i+2) = \sum_{i=1}^{n} i\ (i+1)$$

$$= \sum_{i=1}^{n} i^2 + \sum_{i=1}^{n} i = \frac{1}{3}\ \frac{(n+2)!}{(n-1)!}$$

$$\Rightarrow \sum_{i=1}^{n} i^2 = \tfrac{1}{3}\ n\ (n+1)\ (n+2)\ - \tfrac{1}{2}\ n\ (n+1)$$

$$= n\ (n+1)\ \ \frac{2(n+2)-3}{6}$$

$$= \frac{n(n+1)\ (2n+1)}{6}$$

(ii)

$$\sum_{i=\frac{m(m-1)}{2}+1}^{\frac{m(m+1)}{2}} (2i-1) = \sum_{i=1}^{\frac{m(m+1)}{2}} (2i-1) - \sum_{i=1}^{\frac{m(m-1)}{2}} (2i-1)$$

$$= 2 \left[\frac{\frac{m(m+1)}{2}\ (\frac{m(m+1)}{2} + 1)}{2}\right] - \frac{m(m+1)}{2}$$

$$- 2 \left[\frac{\frac{m(m-1)}{2}\ (\frac{m(m-1)}{2} + 1)}{2}\right] + \frac{m(m-1)}{2}$$

$$= \tfrac{m(m+1)}{2} \left(\tfrac{m(m+1)}{2} + 1\right) - \tfrac{m(m-1)}{2} \left(\tfrac{m(m-1)}{2} + 1\right) + \tfrac{m}{2}(m-1- m-1)$$

$$= \tfrac{m^2(m+1)^2}{4} + \tfrac{m(m+1)}{2} - \tfrac{m^2(m-1)^2}{4} - \tfrac{m(m-1)}{2} - m$$

$$= \tfrac{m^2(m^2+2m+1)}{4} - \tfrac{m^2(m^2-2m+1)}{4}$$

$$= \tfrac{2m^3}{4} + \tfrac{2m^3}{4} = m^3$$

(iii) Racionalicemos el término general:

$$\tfrac{1}{\sqrt{k+1}\ +\ \sqrt{k}} = \tfrac{\sqrt{k+1}-\sqrt{k}}{k+1-k} = (\sqrt{k+1} - \sqrt{k})$$

$$\Rightarrow \ \frac{1}{\sqrt{k(k+1)}} \cdot \frac{1}{(\sqrt{k+1}+\sqrt{k})} \ = \ \frac{\sqrt{k+1} \ - \ \sqrt{k}}{\sqrt{k} \ \sqrt{k+1}} \ = \ \frac{1}{\sqrt{k}} \ - \ \frac{1}{\sqrt{k+1}}$$

$$\Rightarrow \ \sum_{k=1}^{n} \frac{1}{\sqrt{k(k+1)} \ (\sqrt{k+1}+\sqrt{k})} \ = \sum_{k=1}^{n} \left(\frac{1}{\sqrt{k}} \ - \ \frac{1}{\sqrt{k+1}}\right)$$

que es una telescópica

$$= \frac{1}{\sqrt{1}} \ - \ \frac{1}{\sqrt{n+1}} \ = \ (1 - \frac{1}{\sqrt{n+1}})$$

Pregunta 2.

(i)

$\underline{n=1}$:

$$\frac{1}{\sqrt{1}} \qquad \wedge \qquad 2\,(\sqrt{1+1}\ -\ 1)\ =\ 2\,(\sqrt{2}-1)$$

$$=\ 2\,\sqrt{2}-2\ <\ 1$$

pues $\qquad 2\,\sqrt{2}\ <\ 3$

H.I.: para n es cierto.

Pruebo para $n+1$:

$$\sum_{i=1}^{n+1}\frac{1}{\sqrt{i}}\ =\ \sum_{i=1}^{n}\frac{1}{\sqrt{i}}\ +\ \frac{1}{\sqrt{n+1}}\ \geq\ 2\,(\sqrt{n+1}\ -\ 1)\ +\frac{1}{\sqrt{n+1}}$$

por H.I.

Veamos a través de equivalencias si el último término es $\geq\ 2\,(\sqrt{n+2}\ -\ 1)$:

$$2\,(\sqrt{n+1}\ -\ 1)\ +\ \frac{1}{\sqrt{n+1}}\ \geq\ 2\,(\sqrt{n+2}\ -1)$$

$$\Leftrightarrow\ 2\ \left((n+1)\ -\ \sqrt{n+1}\right)\ +\ 1\ \geq\ 2\ \left(\sqrt{n+2}\ \sqrt{n+1}\ -\sqrt{n+1}\right)$$
$$\Leftrightarrow\ 2n+2+1\ \geq\ 2\,\sqrt{n+2}\ \sqrt{n+1}$$
$$\Leftrightarrow 2n+3\ \geq\ 2\,\sqrt{n^2+3n+2}$$
$$\Leftrightarrow 4n^2+12n+9\ \geq\ 4n^2+\ 6n\ +\ 4$$
$$\Leftrightarrow 6n+5\ \geq\ 0\ \text{ que es cierto para } n\ \geq\ 1$$

(ii)

$\underline{n=1}$:

$$\varphi_1(x_0,x_1)=x_0+x_1=\begin{pmatrix}1\\0\end{pmatrix}x_0+\begin{pmatrix}1\\1\end{pmatrix}x_1$$

H.I. : asumo fórmula para $n\geq 1$.

P.d. caso $n+1$:

$$
\begin{aligned}
\varphi_{n+1}(x_0,\cdots,x_{n+1}) &= \varphi_n(x_0,\cdots,x_n)+\varphi_n(x_1,\ \cdots,x_{n+1})\\
&= \sum_{k=0}^{n}\binom{n}{k}x_k+\sum_{k=0}^{n}\binom{n}{k}x_{k+1}\\
&= x_0+\sum_{k=1}^{n}\binom{n}{k}x_k+\sum_{k=0}^{n-1}\binom{n}{k}x_{k+1}+x_{n+1}\\
&= x_0+\sum_{k=1}^{n}\left[\binom{n}{k}+\binom{n}{k-1}\right]x_k+x_{n+1}\\
&= x_0+\sum_{k=1}^{n}\left[\binom{n+1}{k}\right]x_k+x_{n+1}\\
&= \sum_{k=0}^{n+1}\binom{n+1}{k}x_k
\end{aligned}
$$

(iii)

Como $A\ \subseteq\ \mathbb{Q}$ y A es infinito pues la familia $\{\frac{1}{3^i}\ /\ i\geq 0\}$ está en biyección con $\mathbb{N}$, luego A es numerable.

Pregunta 3.

(i) <u>*Refleja*</u>: claramente $x \ R \ x$ pues $f^{(0)}(x) = id(x) = x$

<u>Simetría</u>:

Sup. $x \ R \ y \Leftrightarrow \ \exists \ n \ \in \ \mathbb{Z}, \ f^n(x) = y.$

Pero $f^{(n)}$ es invertible y $(f^{(n)})^{-1} = f^{(-n)} \ \Rightarrow \ f^{(-n)} \ (y) = x$

lo que significa que y R x.

<u>*Transitividad*</u> :

Supongamos $x \ R \ y \ \wedge \ y \ R \ z$. Luego $f^{(n)}(x) = \ y$ y $f^{(m)}(y) = z$ para ciertos $n, m \ \in \ \mathbb{Z}$, luego

$$f^{(m)} \ (f^{(n)}(x)) \ = \ z.$$

Pero $f^{(m)} \ \circ \ f^{(n)} = f^{(m+n)}$. En efecto.

Si $m \ < \ 0 \ \wedge \ n \ < \ 0 \ \Rightarrow (m+n) \ < \ 0$ y $f^{(m)} \ \circ \ f^{(n)} = (f^{-1})^{(|m|)} \ \circ \ (f^{-1})^{(|n|)}$

$$= f^{-1(|m|+|n|)}$$
$$= f^{(m+n)}$$

Si $m > 0 \ \wedge \ n > 0 \Rightarrow (m+n) > 0$ y $f^{(m)} \ \circ \ f^{(n)} \ = f^{(m+n)}$

Si $m < 0 \wedge n > 0 \Rightarrow \underbrace{f^{-1} \ \circ \cdots \circ}_{|m|} \ \underbrace{f^{-1} \ \circ}_{id} \ \underbrace{f \cdots \ \circ f}_{n} =$

$$\text{Si } |m| > n \Rightarrow (f^{-1})^{|m|-n} = f^{(m+n)}$$
$$|m| < n \ \Rightarrow \ (f)^{n-|m|} \quad = f^{(m+n)}$$

Análogamente $m > 0$ y $n < 0 \ \Rightarrow \ f^{(m+n)} = f^{(m)} \ \circ \ f^{(m)}$

(ii)

$$[0] = \{x \ \in \ \mathbb{Q} \ / \ f^{(n)} \ (0) = x \text{ para un cierto } n \ \in \ \mathbb{Z} \ \}$$
$$= \{ \ f^{(n)} \ (0) \ / \ n \ \in \ \mathbb{Z} \ \}$$
$$= \{ \ 0 \ / \ n \ \in \ \mathbb{Z}\} = \ \{0\}$$
$$[1] = \ \{f^{(n)} \ (1) \ / \ n \ \in \ \mathbb{Z} \ \}$$
$$= \ \{p^n \ / \ n \ \in \ \mathbb{Z}\}$$