PAUTA CONTROL RECUPERATIVO 1 - MA11A-ALGEBRA

(1999)

Pregunta 1.

(a)

(i)

$$\sum_{i=1}^{n} 2^{i+1} \cdot \frac{i}{(i+1)(i+2)} = \sum_{i=1}^{n} 2^{i+1}\left[\frac{-1}{i+1} + \frac{2}{i+2}\right]$$
$$= \sum_{i=1}^{n}\left[\frac{2^{i+2}}{i+2} - \frac{2^{i+1}}{i+1}\right]$$

$$\text{Telescópica} \rightarrow = \frac{2^{n+2}}{n+2} - \frac{2^2}{2} = \frac{2^{n+2}}{n+2} - 2$$

(ii)

$$\sum_{i=1}^{2n} (-1)^i i = \sum_{j=1}^{n}(2j) - \sum_{j=1}^{n}(2j-1)$$
$$= 2\frac{n(n+1)}{2} - 2\frac{n(n+1)}{2} + n$$
$$= n$$

(iii) Nuestra hipótesis de inducción en $n \in I\!N$ es:

H.I.: "$\forall k \in \{0, \cdots, n\} \qquad a(n,k) = \binom{n}{k}2^{n-k}$"

Probémosla en n=0: *i.e.* $a(0,0) = 1 = \binom{0}{0}2^{0-0}$, que es evidente.

Asumamos H.I. cierta hasta n y probemos (n+1):

Sea $k = 0$ entonces $a(n+1,0) = 2 \cdot a(n,0) = 2 \cdot \binom{n}{0}2^{n-0}$

$$= \binom{n}{0}2^{n+1} = \binom{n+1}{0}2^{n+1-0}$$

$k = n+1$ entonces $a(n+1, n+1) = a(n,n) = \binom{n}{n} 2^{n-n} = 1$

$$= \binom{n+1}{n+1} \cdot 2^{(n+1)-(n+1)}$$

Sea $k \in \{1, \cdots, n\}$, entonces $a(n+1, k) = a(n, k-1) + 2a(n,k)$

$$= \binom{n}{k-1} 2^{n-(k-1)} + 2 \cdot \binom{n}{k} 2^{n-k}$$

$$= 2^{n-k+1}\left[\binom{n}{k-1} + \binom{n}{k}\right] = \binom{n+1}{k} 2^{(n+1)-k}$$

Pregunta 2.

(a)

$$z = a + bi, \qquad |z|^2 = a^2 + b^2 = 1$$
$$|z+1|^2 = (a+1)^2 + b^2 = 1 \Rightarrow a^2 + b^2 + 2a + 1 = 1$$

$$\Rightarrow 2 \quad + 2a = 1 \Rightarrow a = -\frac{1}{2}$$
$$\Rightarrow \frac{1}{4} + b^2 = 1 \Rightarrow b = \pm\frac{\sqrt{3}}{2}$$

$$\Rightarrow z = \frac{-1}{2} + i\frac{\sqrt{3}}{2} \ o \ z = \frac{-1}{2} - i\frac{\sqrt{3}}{2} \ i.e.: z = e^{i\frac{2\pi}{3}} \ o \ z = e^{-i\frac{2\pi}{3}}$$

$$\Rightarrow z^3 = (cos\frac{2\pi}{3} + isen\frac{2\pi}{3})^3 \ o \ z^3 = cos(\frac{-2\pi}{3}) + isen(\frac{2\pi}{3})^3$$
$$= cos 2\pi + isen 2\pi = 1 \qquad = cos(-2\pi) + isen(-2\pi) = 1$$

$$\Rightarrow z^3 = 1$$

(b)

(i) $z_1 = a + bi, z_2 = c + di, \overline{z}_1 = a - bi$

$$\begin{aligned}\Rightarrow |1\cdot \overline{z}_1 - z_2|^2 - |z_1 - z_2|^2 &= |1 - [(ac+bd) + i(da - bc)]^2 - |(a-c) + i(b-d)|^2 \\ &= (1 - ac - bd)^2 + (da - bc)^2 - (a-c)^2 - (b-d)^2 \\ &= 1 + a^2c^2 + b^2d^2 - 2ac - 2bd + 2abcd + d^2a^2 + b^2c^2 \\ &\quad - 2abcd - a^2 - c^2 + 2ac - b^2 - d^2 + 2bd \\ &= 1 + a^2(c^2 + d^2) + b^2(d^2 + c^2) - (a^2 + b^2) - (c^2 + d^2) \\ &= 1 + (a^2 + b^2)(c^2 + d^2) - (a^2 + b^2) - (c^2 + d^2) \\ &= (1 - (a^2 + b^2))(1 - (c^2 + d^2)) \\ &= (1 - |z_1|^2)(1 - |z_2|^2)\end{aligned}$$

(ii) Si $|z_1| < 1 \ y \ |z_2| < 1 \Rightarrow 0 < (1 - |z_1|^2) < 1 \ y \ 0 < (1 - |z_2|^2) < 1$

$\Rightarrow 0 < |1 - \overline{z}_1 \cdot z_2|^2 - |z_1 - z_2|^2 < 1$

$\Rightarrow 0 < \underbrace{(|1 - \overline{z}_1 \cdot z_2| - |z_1 - z_2|)}_{A-B} 1 - \underbrace{(|\overline{z}_1 \cdot z_2| + |z_1 - z_2|}_{A+B)} < 1$

$\Rightarrow$ Como $A + B > 0$ y $(A + B) \cdot (A - B) > 0 \Rightarrow (A - B) > 0$.

Ademas $A - B \leq A + B \Rightarrow (A - B) < 1$ pues sino$(A - B) > 1, (A + B) > 1$

y en ese caso $(A - B) \cdot (A + B) > 1$

Pregunta 3.

(i)

$-f(0) = 0 \in \mathbb{Z}$ pues f morfismo y f neutro $=$ neutro.

$-$ asumo para $n \in \mathbb{N}, f(n) \in \mathbb{Z}$.

$-$ $\underline{n+1}: f(n+1) = f(n) + f(1)$ y $f(n) \in \mathbb{Z}$ por H.I.

$f(1) \in \mathbb{Z}$ por hipótesis

$$\Rightarrow f(n+1) \in \mathbb{Z}$$

(ii)

$Si\ n \in \mathbb{Z}\ y\ n \geq 0 \Rightarrow$ por $(i) f(n) \in \mathbb{Z}$.

$n < 0 \Rightarrow n = -|n|$ y como f es morfismo

entre grupos $f(n) = f(-|n|) = -f(|n|)$.

Pero de$(i) f(|n|) \in \mathbb{Z}$, luego$f(n) = -f(|n|) \in \mathbb{Z}$

(iii)

$f(\frac{0}{q}) = f(0) = 0$ (idem (i))

H.I.: $f(\frac{p}{q}) = pf(\frac{1}{q})$ para $p \in \mathbb{N}$.

p + 1:

$$f(\frac{p+1}{q}) = f(\frac{p}{q} + \frac{1}{q}) = f(\frac{p}{q}) + f(\frac{1}{q}) = pf(\frac{1}{q}) + f(\frac{1}{q})$$

$$= (p+1)f(\frac{1}{q})$$

(iv)

Si $p \in \mathbb{Z}$: $\quad p \geq 0 \Rightarrow f(\dfrac{p}{q}) = pf(\dfrac{1}{q})$

$$p < 0 \Rightarrow p = -|p| \Rightarrow f(\frac{p}{q}) = f(-\frac{|p|}{q}) = -f(\frac{|p|}{q})$$

$$= -|p| \cdot f(\frac{1}{q}) = p \cdot f(\frac{1}{q})$$

(v)

$$\text{Calculemos } f(\frac{1}{q}): \text{ como } 1 = \frac{q}{q} \Rightarrow f(1) = f(\frac{q}{q}) = q \cdot f(\frac{1}{q})$$
$$\Rightarrow f(\frac{1}{q}) = f(1) \cdot \frac{1}{q}$$
$$\Rightarrow f(\frac{p}{q}) = p \cdot f(\frac{1}{q}) = f(1) \cdot \frac{p}{q}$$

(vi)

$$f^n : Q \to Q$$
$$x \to f^n(x) = (f(1))^n \cdot x$$

Entonces probemos que $(V, +)$ es subgrupo de $(Q, +)$ usando forma compacta. Sean $\frac{p_1}{q_1} \frac{p_2}{q_2} \in V$ luego $(f(1))^{n_1} \cdot \frac{p_1}{q_1} \in \mathbb{Z}$ y $(f(1))^{n_2}\ \frac{p_2}{q_2} \in \mathbb{Z}$.

Luego, como $f(1) \in \mathbb{Z}$ si tomo $n = max(n_1, n_2)$,

$$(f(1))^n \cdot (\frac{p_1}{q_1} - \frac{p_2}{q_2}) = f(1)^n \frac{p_1}{q_1} - f(1)^n \frac{p_2}{q_2}$$
$$= f(1)^{n-n_1} f(1)^{n_1} \frac{p_1}{q_1} - f(1)^{n-n_2} f(1)^{n_2} \frac{p_2}{q_2}$$

$$\Rightarrow (\frac{p_1}{q_1} - \frac{p_2}{q_2}) \in V \Rightarrow (V, +) \text{ es subgrupo de } (Q, +).$$