Pauta Control 2 Cálculo en Varias Variables

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P1. a) $z = f(x,y) = x^2 + y^2$.

$$\nabla f(x,y) = (2x, 2y) \Rightarrow \nabla f(0,-\pi) = (0,-2\pi)$$

Vector normal al plano tangente:$\vec{n} = (\frac{\partial f}{\partial x}(0,-\pi), \frac{\partial f}{\partial y}(0,-\pi) - 1)$. Como el plano pasa por el punto $(0,-\pi,\pi^2)$ su ecuación es

$$\vec{n} \cdot ((x,y,z) - (0,-\pi,\pi^2)) = 0$$

o equivalentemente

$$z = -\pi^2 - 2\pi y$$

b) $\sigma(t) = (tsent, tcost, t^2)$. Sea $t \in I\!R$ entonces

$$(tsent)^2 + (tcost)^2 = t^2\{sen^2 t + cos^2 t\} = t^2$$

es decir $\sigma(t) \in S$ $\forall t$. Además $\sigma(\pi) = (0,-\pi,\pi^2)$ es decir, la curva pasa por el punto $(0,-\pi,\pi^2)$ para $t = \pi$.

$$\frac{d\sigma}{dt}(t) = (sent + tsent, cost - tsint, 2t)$$

$\frac{d\sigma}{dt}(\pi) = (-\pi, -1, 2\pi) = v$, vector tangente a la curva en $(0,-\pi,\pi^2)$

c) Queremos encontrar $(x,y,z) \in S$ que minimize la cantidad

$$\|(x,y,z) - (3,0,0)\|.$$

Esto se plantea, de manera equivalente, como el siguiente problema de minimización

$$Min \;\; x^2 - 6x + y^2 + z^2$$
$$s.a. \;\; x^2 + y^2 = z.$$

Definimos la función Lagrangeana

$$L(x,y,z,\lambda) = x^2 - 6x + y^2 + z^2 - \lambda(x^2 + y^2 - z)$$

$$9 - 6 + 1 + 1 = 5$$

Entonces debemos resolver el siguiente sistema de ecuaciones $\nabla L_{(x,y,z,\lambda)} = 0$, esto es

$$\begin{aligned} 2x - 6 &= 2x\lambda && (1)\\ 2y &= 2y\lambda && (2)\\ 2z &= -\lambda && (3)\\ x^2 + y^2 &= z && (4) \end{aligned}$$

Si $y \neq 0$, entonces $\lambda = 1$, lo que contradice (1). Por lo tanto $y = 0$. Reemplazando (3) y (4) en (1) obtenemos la ecuación $2x^3 + x = 3$. Como $2x^3 + x - 3 = (x-1)(2x^2 + 2x + 3)$, la única solución real de la ecuación es $x = 1$ y por (4) entonces $z = 1$. De esta forma el punto que minimiza la distancia es $(1, 0, 1)$ y el valor es $\sqrt{5}$.

P2. a) Expansión de Taylor de orden 2 en torno a x_0:

$$f(x_0 + h) = f(x_0) + \nabla f(x_0) \cdot h + h^t Hf(x_0) h + R_2(h)$$

Como x_0 es mínimo local, $\nabla f(x_0) = \vec{0}$, entonces

$$f(x_0 + h) = f(x_0) + h^t Hf(x_0) h + R_2(h).$$

x_0 es mínimo local $\Leftrightarrow \exists r > 0$ t.q. $\forall h \in B(\vec{0}, r), f(x_0 + h) \geq f(x_0)$. Tomando un tal h:

$$0 \leq f(x_0 + h) - f(x_0) = h^t Hf(x_0) h + R_2(h)$$

Sea v un vector unitario y sea $t \in (0, r)$. Así, $\|tv\| = t\|v\| = t < r$. Usando $h = tv$:

$$\begin{aligned} 0 &\leq t^2 v^t Hf(x_0) v + R_2(tv) \quad / \cdot \frac{1}{t^2}\\ 0 &\leq v^t Hf(x_0) v + \frac{R_2(tv)}{t^2} \quad (*) \end{aligned}$$

Como $f \in C^2, \lim\limits_{t\to 0} \frac{R_2(tv)}{\|tv\|^2} = 0$ tomando $\lim\limits_{t\to 0}$ en (*) se obtiene $0 \leq v^t Hf(x_0) v, \forall v$ vector unitario. Para un h cualquiera, $\frac{h}{\|h\|}$ es unitario, luego $0 \leq h^t Hf(x_0) h \cdot \frac{1}{\|h\|^2}$ y entonces $0 \leq h^t Hf(x_0) h$. ■

b) Si x_0 es un mínimo local de u entonces es fácil ver que

$$e_i^t Hf(x_0) e_i = (Hf(x_0))_{ii} = \frac{\partial^2 f}{\partial x_i^2}(x_0) \geq 0 \quad \forall i.$$

Luego sumando

$$\sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}(x_0) \geq 0.$$

Pero esto es imposible pues

$$\Delta f(x_0) = -1 < 0.$$

Por lo tanto x_0 no puede ser un mínimo local, $\forall x_0 \in I\!R^n$.

P3. a) Consideramos la función $v(x,y) = u(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2})$. Tenemos

$$\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x_1} \cdot \frac{\partial x_1}{\partial x} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial x}$$

$$\begin{aligned}\frac{\partial^2 v}{\partial x^2} =& \frac{\partial}{\partial x}(\frac{\partial u}{\partial x_1} \cdot \frac{\partial x_1}{\partial x} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial x}) \\ =& \frac{\partial^2 u}{\partial x_1^2}(\frac{\partial x_1}{\partial x})^2 + 2\frac{\partial^2 u}{\partial x_1 \partial x_2}(\frac{\partial x_1}{\partial x}\frac{\partial x_2}{\partial x}) + \frac{\partial^2 u}{\partial x_2^2}(\frac{\partial x_2}{\partial x})^2 + \\ & \frac{\partial u}{\partial x_1}(\frac{\partial^2 x_1}{\partial x^2}) + \frac{\partial u}{\partial x_2}(\frac{\partial^2 x_2}{\partial x^2})\end{aligned}$$

De manera completamente análoga:
$\frac{\partial^2 v}{\partial y^2} = \frac{\partial^2 u}{\partial x_1^2}(\frac{\partial x_1}{\partial y})^2 + 2\frac{\partial^2 u}{\partial x_1 \partial x_2}(\frac{\partial x_1}{\partial y}\frac{\partial x_2}{\partial y}) + \frac{\partial^2 u}{\partial x_2^2}(\frac{\partial x_2}{\partial y})^2 + \frac{\partial u}{\partial x_1}(\frac{\partial^2 x_1}{\partial y^2}) + \frac{\partial u}{\partial x_2}(\frac{\partial^2 x_2}{\partial y^2})$
En consecuencia

$$\begin{aligned}\Delta v ==& \frac{\partial^2 u}{\partial x_1^2}\{(\frac{\partial x_1}{\partial x})^2 + (\frac{\partial x_1}{\partial y})^2\} + \frac{\partial^2 u}{\partial x_2^2}\{(\frac{\partial x_2}{\partial x})^2 + (\frac{\partial x_2}{\partial y})^2\} \\ & + 2\frac{\partial^2 u}{\partial x_1 \partial x_2}(\frac{\partial x_1}{\partial x}\frac{\partial x_2}{\partial x} + \frac{\partial x_1}{\partial y}\frac{\partial x_2}{\partial y}) \\ & + \frac{\partial u}{\partial x_1}(\frac{\partial^2 x_1}{\partial x^2} + \frac{\partial^2 x_1}{\partial y^2}) + \frac{\partial u}{\partial x_2}(\frac{\partial^2 x_2}{\partial x^2} + \frac{\partial^2 x_2}{\partial y^2}).\end{aligned}$$

Por otra parte

$$\begin{aligned}\frac{\partial x_1}{\partial x} &= \frac{y^2 - x^2}{(x^2+y^2)^2}, & \frac{\partial x_1^2}{\partial x^2} &= \frac{(x^2 - 3y^2)}{(x^2+y^2)^3}2x \\ \frac{\partial x_2}{\partial x} &= \frac{-2xy}{(x^2+y^2)^2}, & \frac{\partial^2 x_2}{\partial x^2} &= \frac{2y(3x^2 - y^2)}{(x^2+y^2)^3}\end{aligned}$$

Y de manera completamente análoga

$$\begin{aligned}\frac{\partial x_1}{\partial y} &= \frac{-2xy}{(x^2+y^2)^2}, & \frac{\partial^2 x_1}{\partial y^2} &= \frac{2x(3y^2 - x^2)}{(x^2+y^2)^3} \\ \frac{\partial x_2}{\partial y} &= \frac{(x^2 - y^2)}{(x^2+y^2)^2}, & \frac{\partial^2 x_2}{\partial y^2} &= \frac{(y^2 - 3x^2)}{(x^2+y^2)^3}2y\end{aligned}$$

De lo anterior uno obtiene que

$$\Delta v = \frac{1}{(x^2+y^2)^2}\Delta u.$$

De donde se obtiene la conclusión deseada.

b) $f(x,y,z) = log(x^2+y^2+z^2) \quad \vec{x}_0 = (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$

$$f(x_0+h) = f(\vec{x}_0) + \Delta f(\vec{x}_0)(h) + \frac{(h + Hf(\vec{x}_0)(h)}{2} + resto$$

$$Df(x_0) = \left(\frac{2x}{x^2+y^2+z^2}, \frac{2y}{x^2+y^2+z^2}, \frac{2z}{x^2+y^2+z^2}\right)$$

$$Hf(x_0) = \begin{bmatrix} \frac{2y^2+2z^2-2x^2}{(x^2+y^2+z^2)^2} & \frac{-4xy}{(x^2+y^2+z^2)^2} & \frac{-4xz}{(x^2+y^2+z^2)^2} \\ \frac{-4xy}{(x^2+y^2+z^2)^2} & \frac{2x^2+2z^2-2y^2}{(x^2+y^2+z^2)^2} & \frac{-4yz}{(x^2+y^2+z^2)^2} \\ \frac{-4xz}{(x^2+y^2+z^2)^2} & \frac{-4yz}{(x^2+y^2+z^2)^2} & \frac{2x^2+2y^2+2z^2}{(x^2+y^2+z^2)^2} \end{bmatrix}$$

$$Df(\vec{x}_0) = (\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}).$$

$$Hf(x_0) = \begin{bmatrix} \frac{2}{3} & \frac{-4}{3} & \frac{-4}{3} \\ \frac{-4}{3} & \frac{2}{3} & \frac{-4}{3} \\ \frac{-4}{3} & \frac{-4}{3} & \frac{2}{3} \end{bmatrix}$$

$$f(\frac{1}{\sqrt{3}}+h_1, \frac{1}{\sqrt{3}}+h_2, \frac{1}{\sqrt{3}}+h_3) = 0 + (\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix}$$

$$+\frac{1}{2}(h_1,h_2,h_3)\begin{bmatrix} \frac{2}{3} & \frac{-4}{3} & \frac{-4}{3} \\ \frac{-4}{3} & \frac{2}{3} & \frac{-4}{3} \\ \frac{-4}{3} & \frac{-4}{3} & \frac{2}{3} \end{bmatrix}\begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix}$$

$$= \frac{2}{\sqrt{3}}x + \frac{2}{\sqrt{3}}y + \frac{2}{\sqrt{3}}z - \frac{6}{\sqrt{3}} + \frac{1}{2}\left[\frac{2}{3}x - \frac{4}{3}y - \frac{4}{3}z + \frac{6}{\sqrt{3}}, \frac{2}{3}y - \frac{4}{3}x - \frac{4}{3}z + \right]$$

$$\left.\frac{6}{2\sqrt{3}}, \frac{2}{3}z - \frac{4}{3}x - \frac{4}{3}y + \frac{6}{2\sqrt{3}}\right] \cdot \begin{pmatrix} x - \frac{1}{\sqrt{3}} \\ y - \frac{1}{\sqrt{3}} \\ z - \frac{1}{\sqrt{3}} \end{pmatrix}$$

$$= \frac{2}{\sqrt{3}}(h_1+h_2+h_3) + \frac{1}{3}(h_1,h_2,h_3)\begin{bmatrix} 1 & -2 & -2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{bmatrix}\begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix}$$

$$= \frac{2}{\sqrt{3}}(h_1+h_2+h_3) + \frac{1}{3}(h_1^2 + h^2 2 + h^2 3 - 4h_1h_2 - 4h_2h_3 - 4h_1h_3)$$